ON THE NEWTON POLYGONS OF TWISTED L-FUNCTIONS OF BINOMIALS

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Abstract. Let χ be an order c multiplicative character of a finite field and $f(x) = x^d + \lambda x^e$ a binomial with $(d, e) = 1$. We study the twisted classical and T-adic Newton polygons of f. When $p > (d - e)(2d - 1)$, we give a lower bound of Newton polygons and show that they coincide if p does not divide a certain integral constant depending on p mod cd.

We conjecture that this condition holds if p is large enough with respect to c, d by combining all known results and the conjecture given by Zhang-Niu. As an example, we show that it holds for $e = d - 1$.

Contents

1. Introduction

1.1. **Background.** Fix a rational prime p. For $q = p^a$ a power of p, denote by $\mathbb{F}_q$ the finite field with q elements, $\mathbb{Q}_q$ the unramified extension of $\mathbb{Q}_p$ of degree a and $\mathbb{Z}_q$ its ring of integers. Let $f(x) \in \mathbb{F}_q[x]$ be a polynomial of degree d with Teichmüller lifting $\hat{f}(x) \in \mathbb{Z}_q[x]$. Let $\chi : \mathbb{F}_q^\times \to \mathbb{C}_p^\times$ be a multiplicative character and $\omega : \mathbb{F}_q^\times \to \mathbb{Z}_q^\times$ the Teichmüller lifting. Then we can write $\chi = \omega^{-u}$ for some $0 \le u \le q - 2$.

For a non-trivial additive character $\psi_m : \mathbb{Z}_p \to \mathbb{C}_p^\times$ of order p^m, define the twisted L-function

$$L_u(s, f, \psi_m) = \exp\left(\sum_{k=1}^{\infty} S_{k,u}(f, \psi_m)\frac{s^m}{m}\right),$$

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where $S_{k,u}(f,\psi_m)$ is the twisted exponential sum

$$S_{k,u}(f,\psi_m) = \sum_{x\in\mathbb{F}_{q^k}^\times} \psi_m\left(\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}\left(\hat{f}(\hat{x})\right)\right)\omega^{-u}\left(\mathrm{Nm}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)\right).$$

If $p\nmid d$, then $L_u(s,f,\psi_m)$ is a polynomial of degree $p^{m-1}d$ by Adolphson-Sperber [AS87, AS91, AS93], Li [Li99], Liu-Wei [LW07] and Liu [Liu07].

We will use the twisted T-adic exponential sums developed by Liu-Wan [LW09] and Liu [Liu02, Liu09]. Define the twisted T-adic L-function

$$L_u(s,f,T) = \exp\left(\sum_{k=1}^{\infty} S_{k,u}(f,T)\frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_q[\![T]\!][\![s]\!],$$

where $S_{k,u}(f,T)$ is the twisted T-adic exponential sum

$$S_{k,u}(f,T) = \sum_{x\in\mathbb{F}_{q^k}^\times} (1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(\hat{f}(\hat{x}))}\omega^{-u}\left(\mathrm{Nm}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x)\right).$$

Then $L_u(s,f,\psi_m) = L_u(s,f,\pi_m)$ where $\pi_m = \psi_m(1)-1$.

Denote by

$$C_u(s,f,T) = \prod_{j=0}^{\infty} L_u(q^j s,f,T) \in 1 + s\mathbb{Z}_q[\![T]\!][\![s]\!]$$

the characteristic function, which is T-adic entire in s. Then

$$L_u(s,f,T) = C_u(s,f,T)C_u(qs,f,T)^{-1}.$$

Since the $\pi_m^{a(p-1)}$-adic Newton polygon of $C_u(s,f,\pi_m)$ does not depend on the choice of ψ_m, we denote it by $\mathrm{NP}_{u,m}(f)$. Denote by $\mathrm{NP}_{u,T}(f)$ the $T^{a(p-1)}$-adic Newton polygon of $C_u(s,f,T)$. As shown in [LW09] and [Liu07], $\mathrm{NP}_{u,m}(f)$ lies over the infinity u-twisted Hodge polygon $H^{\infty}_{[0,d],u}$, which has slopes

$$\frac{n}{d} + \frac{1}{bd(p-1)}\sum_{k=1}^{b} u_k,\ \ n\in\mathbb{N}. \tag{1.1}$$

If we write $0\le s_0\le\cdots\le s_{p^{m-1}d-1}\le 1$ the q-adic slopes of $L_u(s,f,\pi_m)$, then the q-adic slopes of $C_u(s,f,\pi_m)$ are

$$j+s_i,\quad 0\le i\le p^{m-1}d-1, j\in\mathbb{N}.$$

That's to say, the $\pi_m^{a(p-1)}$-adic Newton polygon of $L_u(s,f,\pi_m)$ is the restriction of $\mathrm{NP}_{u,m}(f)$ on $[0,p^{m-1}d]$, and it determines $\mathrm{NP}_{u,m}(f)$.

The prime p is required large enough in the following results. When $\chi=\omega^{-u}$ is trivial, in [Zhu14] and [LLN09], they gave a lower bound of the Newton polygons. They defined a polynomial on the coefficients of f, called Hasse polynomial. If the Hasse polynomial is nonzero, then the Newton polygons coincide this lower bound.

Assume that $f(x)=x^d+\lambda x^e$ is a binomial. Since the exponential sums can be transformed to the twisted case when d and e are not coprime, we assume $(d,e)=1$ in this paper. When $u=0$, we list the known cases here.

- $p\equiv 1 \bmod d$, it's well-known that the Newton polygons coincides the Hodge polygon.
- $e=1$, see [Yan03, §1, Theorem], [Zhu14, Theorem 1.1] and [OY16, Theorem 1.1].

- $e = d-1, p \equiv -1 \bmod d$, see [OZ16].
- $e = 2, p \equiv 2 \bmod d$, see [ZN21].

For arbitrary u, Liu-Niu [LN11] obtained the Newton polygons when $e = 1$. Zhang-Niu [ZN21] also give a conjectural description of the Newton polygons when $p \equiv e \bmod d$.

1.2. **Notations.** We list the notations we will use.

- i, j, v, w, k, ℓ, n indices.
- $f(x) = x^d + \lambda x^e \in \mathbb{F}_q[x]$ a binomial with $d > e \geq 1, (d, e) = 1, \lambda \neq 0$.
- $\omega^{-u} : \mathbb{F}_q^\times \to \mathbb{C}_p^\times$, where ω is the Teichmüller lifting and $0 \leq u \leq q-2$.
- $H^\infty_{[0,d],u}$, the infinity u-twisted Hodge polygon with slopes in (1.1).
- $c = \frac{q-1}{(q-1,u)}$ the order of ω^{-u}, then $u = \frac{(q-1)\mu}{c}$ for some $(\mu, c) = 1$.
- $P_{u,e,d}$ a polygon with slopes $w(i)$, defined in (1.2).
- b the least positive integer such that $p^b u \equiv u \bmod (q-1)$ (equivalently, $p^b \equiv 1 \bmod c$).
- $0 \leq u_i \leq p-1$ such that $u = u_0 + u_1 p + \cdots + u_{a-1}p^{a-1}$, $u_i = u_{b+i}$.
- $\overline{x}$ the minimal non-negative residue of x modulo d.
- δ_P takes value 1 if P happens; 0 if P does not happen.
- $I_n = \{1, \ldots, n\}, I_n^* = \{0, 1, \ldots, n\}$.
- S_n (resp. S_n^*) the set of permutations of I_n (resp. I_n^*).
- $C_{t,n}$ the minimum of $\sum_{i=0}^n \overline{e^{-1}(pi - \tau(i) + t)}$ for $\tau \in S_n^*$ and $S_{t,n}^\circ$ the set of $\tau \in S_n^*$ such that the summation reaches minimal. Set $C_{t,-1} = 0$ for convention.
- $R_{i,\alpha} = \overline{e^{-1}(pi + \alpha)}$, $r_{i,\alpha} = \overline{e^{-1}(t - \alpha - i)}$, see Proposition 2.1. We will drop the subscript α if there is no confusion.
- $\mathbf{C}_{t,n,\alpha}$ the maximal size of $\left\{i \in I_n^* \mid R_{i,\alpha} + r_{\tau(i),\alpha} \geq d\right\}$ for $\tau \in S_n^*$. We will drop the subscript α if there is no confusion.
- $y_{t,i}^\tau = \overline{e^{-1}(pi - \tau(i) + t)}$, $x_{t,i}^\tau = d^{-1}(pi - \tau(i) + t - ey_{t,i}^\tau)$ the unique solution of $dx + ey = pi - \tau(i) + t$ with $0 \leq y \leq d-1$.
- $h_{n,k}, h_{u,e,d}$ the Hasse numbers defined in (1.3).
- $\mathbf{p}$ the minimal non-negative residue of p modulo cd.
- $H_{\mu,c,\mathbf{p},e,d} \in \mathbb{Z}$ a constant defined in (3.1).
- $E(X)$ the p-adic Artin-Hasse series, see (2.1).
- π a T-adic uniformizer of $\mathbb{Q}_p[\![T]\!]$ given by $E(\pi) = 1 + T$, with a fixed $d(q-1)$-th root $\pi^{\frac{1}{d(q-1)}}$.
- $E_f(X)$, see (2.2).
- $M_u = \frac{u}{q-1} + \mathbb{N}$.
- $\mathcal{L}_u$ a Banach space, see (2.3).
- $\mathcal{B}_u$ a subspace of $\mathcal{L}_u$, see (2.4).
- $\mathcal{B} = \mathcal{B}_u \oplus \mathcal{B}_{pu} \oplus \cdots \oplus \mathcal{B}_{p^{b-1}u}$.
- $\psi : \mathcal{L}_u \to \mathcal{L}_{p^{-1}u}$ defined as $\psi\left(\sum_{v \in M_u} b_v X^v\right) = \sum_{v \in M_{p^{-1}u}} b_{pv} X^v$.
- $\sigma \in \mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ the Frobenius, which acts on $\mathcal{L}_u$ via the coefficients.
- $\Psi = \sigma^{-1} \circ \psi \circ E_f : \mathcal{B}_u \to \mathcal{B}_{p^{-1}u}$ the Dwork's T-adic semi-linear operator.
- c_n the coefficients of $\det(1 - \Psi s \mid \mathcal{B})$, see (2.6).
- $s_k \equiv p^k u \bmod q-1$ with $0 \leq s_k \leq q-2$.
- $\Gamma = \left(\gamma_{(v, \frac{s_k}{q-1}+i),(w, \frac{s_\ell}{q-1}+j)}\right)$ the matrix coefficient of Ψ on $\mathcal{B}$, see (2.7).
- $\Gamma^{(k)}$ the sub-matrix of Γ defined in (2.7).

- $A^{(k)} = A \cap \Gamma^{(k)}$ the sub-matrix of a principal minor A of Γ.
- $\mathcal{A}_n$ the set of all principal minor A of order bn, such that every $A^{(k)}$ has order n.
- $\phi(n) \in \mathbb{N} \cup \{+\infty\}$ the minimal $x+y$ where $dx+ey=n, x, y \in \mathbb{N}$.
- $\gamma_{(\frac{s_k}{q-1}+i, \frac{s_\ell}{q-1}+j)}$, see (2.9).
- $(x)_{[n]} := x(x-1)\cdots(x-n+1), (x)_{[0]} := 1$ the falling factorial.

1.3. **Main results.** In this paper, we give an explicit lower bound of Newton polygons of twisted L-functions of binomial $f(x) = x^d + \lambda x^e$. We reduce the Hasse polynomial to a certain integer (3.1). Then $p > (d-e)(2d-1)$ does not divide this constant, if and only if this lower bound coincides the Newton polygons. Finally, we show that this condition holds for $e = d-1$.

Denote by $P_{u,e,d}$ the polygon such that

$$P_{u,e,d}(n) = \frac{n(n-1)}{2d} + \frac{1}{bd(p-1)} \sum_{k=1}^{b} \big(nu_k + (d-e)C_{u_k,n-1}\big), \ n \in \mathbb{N}. \tag{1.2}$$

Denote by $w(n) = P_{u,e,d}(n+1) - P_{u,e,d}(n)$. Then

$$w(n) = \frac{n}{d} + \frac{1}{bd(p-1)} \sum_{k=1}^{b} \big(u_k + (d-e)(C_{u_k,n} - C_{u_k,n-1})\big).$$

This polygon lies above the Hodge polygon $H^{\infty}_{[0,d],u}$ with same points at $d\mathbb{Z}$, and $w(n+d) = 1 + w(n)$. Moreover, we have $w(n) \leq w(n+1)$ if $p > (d-e)(2d-1)$. See Proposition 2.1.

Theorem 1.1. *Assume that $p > (d-e)(2d-1)$. Then $\mathrm{NP}_{u,T}(f)$ lies above $P_{u,e,d}$. As a corollary, $\mathrm{NP}_{u,m}(f)$ lies above $P_{u,e,d}$.*

Define

$$h_{n,k} := \sum_{\tau \in S^{\circ}_{u_k,n}} \mathrm{sgn}(\tau) \prod_{i=0}^{n} \frac{1}{x^{\tau}_{u_k,i}! y^{\tau}_{u_k,i}!}, \quad h_{u,e,d} := \prod_{n=0}^{d-2} \prod_{k=1}^{b} h_{n,k}. \tag{1.3}$$

Theorem 1.2. *Assume that $p > (d-e)(2d-1)$. Then*

$$\mathrm{NP}_{u,m}(f) = \mathrm{NP}_{u,T}(f) = P_{u,e,d} \tag{1.4}$$

holds if and only if $h_{u,e,d} \in \mathbb{Z}_p^{\times}$, if and only if $p \nmid H_{\mu,c,\mathbf{p},e,d}$.

Here $H_{\mu,c,\mathbf{p},e,d} \in \mathbb{Z}$ is a constant defined in (3.1) and $\mathbf{p}$ is the minimal positive residue of p modulo cd. Thus we have the following corollary.

Corollary 1.3. *Assume that (1.4) holds for*

$$a, m, p, f(x) = x^d + \lambda x^e \in \mathbb{F}_{p^a}[x], u = \frac{(p^a-1)\mu}{c},$$

where $b \mid a, \lambda \neq 0$ and $p > (d-e)(2d-1)$. Then

(1) *$H_{\mu,c,\mathbf{p},e,d} \neq 0$.*
(2) *For any*

$$a', m', p', f'(x) = x^d + \lambda' x^e \in \mathbb{F}_{p'^{a'}}[x], u' = \frac{(p'^{a'}-1)\mu}{c},$$

where $b \mid a, \lambda \neq 0$ and $p' > (d-e)(2d-1)$, we have (1.4) if $p' \equiv p \bmod cd$ and $p' > H_{\mu,c,\mathbf{p},e,d}$.

(3) *As $p' \equiv p \bmod cd$ tends to infinity, the polygons* $\mathrm{NP}_{u,m}(f)$ *and* $\mathrm{NP}_{u,T}(f)$ *tend to* $H^{\infty}_{[0,d],u}$*, which only depends on* $\mu, c, \mathbf{p}, d$*.*

The following result extends [OZ16], as they considered the untwisted case with an additional condition $p \equiv -1 \bmod d$.

Theorem 1.4. *Assume that $e = d-1$. We have* $\mathrm{NP}_{u,m}(f) = \mathrm{NP}_{u,T}(f) = P_{u,e,d}$ *if* $p > c(d^2 - d + 1)$*.*

We give the following conjecture, which generalizes the conjecture in [ZN21]. Note that $h_{u,e,d}$ may be zero since $S^{\circ}_{u_k,n}$ may be empty, so we require that p is large with respect to c, as in Corollary 1.3 and Theorem 1.4.

Conjecture 1.5. *If p is large enough with respect to c, d, then* $\mathrm{NP}_{u,m}(f) = \mathrm{NP}_{u,T}(f) = P_{u,e,d}$*.*

2. The lower bound

2.1. The property of the lower bound polygon.

For any integer t, we denote

$$C_{t,n} = \min_{\tau \in S_n^*} \sum_{i=0}^{n} \overline{e^{-1}(pi - \tau(i) + t)}.$$

We set $C_{t,-1} = 0$ for convention. For any integer α, we denote

$$R_{i,\alpha} = \overline{e^{-1}(pi + \alpha)},\ \ r_{i,\alpha} = \overline{e^{-1}(t - \alpha - i)}$$

and

$$\mathbf{C}_{t,n,\alpha} = \max \# \left\{ i \in I_n^* \mid R_{i,\alpha} + r_{\tau(i),\alpha} \geq d \right\}.$$

Proposition 2.1. *(1) For any α, we have*

$$C_{t,n} = \sum_{i=0}^{n} (R_{i,\alpha} + r_{i,\alpha}) - d\mathbf{C}_{t,n,\alpha}.$$

(2) For any α, we have

$$\mathbf{C}_{t,n+d,\alpha} = d - 1 + \mathbf{C}_{t,n,\alpha}, \quad C_{t,n+d} = C_{t,n}.$$

Thus $w(n+d) = 1 + w(n)$ and $P_{u,e,d}(dn) = H^{\infty}_{[0,d],u}(dn)$.

(3) If $p > (d-e)(2d-1)$, we have $w(n) \leq w(n+1)$.

Proof. We omit the subscript α in this proof for convention.

(1) It follows from

$$\overline{e^{-1}(pi - \tau(i) + t)} = R_i + r_{\tau(i)} - d\delta_{R_i + r_{\tau(i)} \geq d}.$$

(2) We have

$$\mathbf{C}_{t,n} = \max_{\tau \in S_n^*} \# \left\{ i \in I_n^* \mid R_i \geq d - r_{\tau(i)} \right\}.$$

Note that

$$\left\{ R_i \mid i \in I_{n+d}^* \right\} = \left\{ R_i \mid i \in I_n^* \right\} \cup \left\{ 0, 1, \ldots, d-1 \right\},$$

$$\left\{ d - r_i \mid i \in I_{n+d}^* \right\} = \left\{ d - r_i \mid i \in I_n^* \right\} \cup \left\{ d, 1, \ldots, d-1 \right\}.$$

We may drop the 0 and d since they do not affect the size. Apple Lemma 2.2 $(d-1)$ times, where $a_0 = b_0 = j$ in j-th time, then we get $\mathbf{C}_{t,n+d} = d - 1 + \mathbf{C}_{t,n}$.

Since
$$\sum_{i=n+1}^{n+d}(R_i+r_i)=2\sum_{j=0}^{d-1}j=d(d-1),$$
we have $C_{t,n+d}=C_{t,n}$. Thus $w(n+d)=1+w(n)$.

Note that $C_{t,n+d}=C_{t,n}$ also holds for $n=-1$. Hence $C_{t,dn-1}=0$ and $P_{u,e,d}(dn)=H^{\infty}_{[0,d],u}(dn)$.

(3) Denote by $\delta=\delta_{R_n+r_n\geq d}$. For any $\tau\in S_n^*$, write $i=\tau(n)$, $j=\tau^{-1}(n)$ and $\tau_1=(ni)\tau$. Then $\tau_1(n)=n$, $\tau_1(j)=i$ and
$$\begin{aligned}&\delta+\#\left\{i\in I_{n-1}^*\mid R_i+r_{\tau_1(i)}\geq d\right\}-\#\left\{i\in I_n^*\mid R_i+r_{\tau(i)}\geq d\right\}\\=&\delta+\delta_{R_j+r_i\geq d}-\delta_{R_j+r_n\geq d}-\delta_{R_n+r_i\geq d}.\end{aligned}$$
If this is -2, then $2d>R_n+r_n+R_j+r_i\geq 2d$, that's impossible. Thus $\delta+\mathbf{C}_{t,n-1}-\mathbf{C}_{t,n}\geq -1$.

Any $\sigma\in S_{n-1}^*$ can be viewed as an element $\sigma_1\in S_n^*$ fixing n. Thus
$$\delta+\#\left\{i\in I_{n-1}^*\mid R_i+r_{\sigma(i)}\geq d\right\}=\#\left\{i\in I_n^*\mid R_i+r_{\sigma_1(i)}\geq d\right\}.$$
and then $\delta+\mathbf{C}_{t,n-1}\leq\mathbf{C}_{t,n}$.

Now
$$\begin{aligned}&C_{t,n}-C_{t,n-1}\\=&R_n+r_n-d(\mathbf{C}_{t,n}-\mathbf{C}_{t,n-1})\\=&\overline{e^{-1}(pn-n+t)}+d(\delta+\mathbf{C}_{t,n-1}-\mathbf{C}_{t,n})\end{aligned}$$
lies in $[-d,d-1]$. Therefore,
$$\begin{aligned}&w(n)-w(n-1)\\=&\frac{1}{d}+\frac{d-e}{bd(p-1)}\sum_{k=1}^{b}(C_{u_k,n}-2C_{u_k,n-1}+C_{u_k,n-2})\\\geq&\frac{1}{d}+\frac{(d-e)(1-2d)}{d(p-1)}\geq 0\end{aligned}$$
since $p>(d-e)(2d-1)$. □

Lemma 2.2. *Let $A=\{a_0,\dots,a_m\}$ and $B=\{b_0,\dots,b_m\}$ be two multi-sets of integers. Assume that $a_0\geq b_0$ and for any $i>0$, $b_i>a_0$ or $b_i\leq b_0$. Then*
$$\max_{\tau\in S_m^*}\#\left\{i\in I_m^*\mid a_i\geq b_{\tau(i)}\right\}=1+\max_{\sigma\in S_m}\#\left\{i\in I_m\mid a_i\geq b_{\sigma(i)}\right\}.$$

Proof. Every permutation in S_n can be viewed as a permutation in S_n^* fixing 0, thus "$\geq$" holds trivially. Write $i=\tau(0)$, $j=\tau^{-1}(0)$ and $\tau_1=(0i)\tau$. Then $\tau_1(0)=0$ and $\tau_1(j)=i$. Thus
$$\begin{aligned}&\#\left\{i\in I_m^*\mid a_i\geq b_{\tau_1(i)}\right\}-\#\left\{i\in I_m^*\mid a_i\geq b_{\tau(i)}\right\}\\=&1+\delta_{a_j\geq b_i}-\delta_{a_j\geq b_0}-\delta_{a_0\geq b_i}.\end{aligned}$$
If this is negative, then $a_0\geq b_i>a_j\geq b_0$, which is impossible. Thus "$\leq$" holds. □

2.2. **The twisted T-adic Dwork's trace formula.** This part is almost the same with [LN11, 2,3]. Denote by

$$E(X) = \exp\left(\sum_{i=0}^{\infty} p^{-i} X^{p^i}\right) = \sum_{n=0}^{\infty} \lambda_n X^n \in \mathbb{Z}_p[\![X]\!] \tag{2.1}$$

the p-adic Artin-Hasse series. Then $\lambda_n = 1/n!$ if $n < p$. Denote by

$$E_f(X) = E(\pi X^d) E(\pi \hat{\lambda} X^e) = \sum_{n=0}^{\infty} \gamma_n X^n. \tag{2.2}$$

Then

$$\gamma_k = \sum \pi^{x+y} \lambda_x \lambda_y \hat{\lambda}^y,$$

where (x, y) runs through non-negative solutions of $dx + ey = k$.

Denote by $M_u = \frac{u}{q-1} + \mathbb{N}$. Define

$$\mathcal{L}_u = \left\{ \sum_{v \in M_u} b_v \pi^{\frac{v}{d}} X^v \;\middle|\; b_v \in \mathbb{Z}_q[\![\pi^{\frac{1}{d(q-1)}}]\!] \right\} \tag{2.3}$$

and

$$\mathcal{B}_u = \left\{ \sum_{v \in M_u} b_v \pi^{\frac{v}{d}} X^v \in \mathcal{L}_u \;\middle|\; \mathrm{ord}_\pi b_v \to +\infty \text{ as } v \to +\infty \right\}. \tag{2.4}$$

Define a map

$$\begin{aligned} \psi : \mathcal{L}_u &\longrightarrow \mathcal{L}_{p^{-1}u} \\ \sum_{v \in M_u} b_v X^v &\longmapsto \sum_{v \in M_{p^{-1}u}} b_{pv} X^v. \end{aligned} \tag{2.5}$$

The power series E_f defines a map on $\mathcal{B}_u$ via multiplication. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ be the Frobenius, which acts on $\mathcal{L}_u$ via the coefficients. Then the Dwork's T-adic semi-linear operator $\Psi = \sigma^{-1} \circ \psi \circ E_f$ sends $\mathcal{B}_u$ to $\mathcal{B}_{p^{-1}u}$. Hence Ψ acts on

$$\mathcal{B} := \bigoplus_{i=0}^{b-1} \mathcal{B}_{p^i u}.$$

We have a linear map

$$\Psi^a = \psi^a \circ \prod_{i=0}^{a-1} E_f^{\sigma^i}(X^{p^i})$$

on $\mathcal{B}$ over $\mathbb{Z}_q[\![\pi^{\frac{1}{d(q-1)}}]\!]$. Since Ψ is completely continuous in the sense of [Ser62], the following determinants are well-defined.

Theorem 2.3. *We have*

$$C_u(s, f, T) = \det\left(1 - \Psi^a s \;\middle|\; \mathcal{B}_u / \mathbb{Z}_q[\![\pi^{\frac{1}{d(q-1)}}]\!]\right).$$

Thus the T-adic Newton polygon of $C_u(s, f, T)$ is the lower convex closure of

$$\left(n, \frac{1}{b} \mathrm{ord}_T(c_{abn})\right), \; n \in \mathbb{N},$$

where

$$\det\left(1 - \Psi s \;\middle|\; \mathcal{B} / \mathbb{Z}_p[\![\pi^{\frac{1}{d(q-1)}}]\!]\right) = \sum_{i=0}^{\infty} (-1)^n c_n s^n. \tag{2.6}$$

Proof. See [LW09, Theorem 4.8], [Liu07], [LLN09, Theorems 2.1, 2.2] and [LN11, Theorems 2.1, 5.3]. □

Write $s_k \equiv p^k u \bmod q-1$ with $0 \le s_k \le q-2$. Then $s_{b-k} = s_{-k} = u_k + u_{k+1}p + \cdots + u_{k+a-1}p^{a-1}$. Let $\xi_1, \ldots, \xi_a$ be a normal basis of $\mathbb{Q}_q$ over $\mathbb{Q}_p$. The space $\mathcal{B}$ has a basis

$$\left\{\xi_v(\pi^{\frac{1}{d}}X)^{\frac{s_k}{q-1}+i}\right\}_{(i,v,k)\in\mathbb{N}\times I_a\times I_b}$$

over $\mathbb{Z}_p[\![\pi^{\frac{1}{d(q-1)}}]\!]$. Let $\Gamma = \left(\gamma_{(v,\frac{s_k}{q-1}+i),(w,\frac{s_\ell}{q-1}+j)}\right)_{\mathbb{N}\times I_a\times I_b}$ be the matrix of Ψ on $\mathcal{B}$ with respect to this basis. Then

$$\Gamma = \begin{pmatrix} 0 & \Gamma^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & \Gamma^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Gamma^{(b-1)} \\ \Gamma^{(b)} & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.7}$$

where

$$\Gamma^{(k)} = \left(\gamma_{(v,\frac{s_{k-1}}{q-1}+i),(w,\frac{s_k}{q-1}+j)}\right)_{\mathbb{N}\times I_a}.$$

Hence we have

$$\det\left(1-\Psi s \;\middle|\; \mathcal{B}/\mathbb{Z}_p[\![\pi^{\frac{1}{d(q-1)}}]\!]\right) = \det(1-\Gamma s) = \sum_{n=0}^{\infty}(-1)^{bn}c_{bn}s^{bn}$$

with $c_n = \sum \det(A)$, where A runs through all principal minors of order n, see [LZ05]. Denote by $A^{(k)} = A \cap \Gamma^{(k)}$ as a minor of $\Gamma^{(k)}$. If A has order bn, but the order of some $A^{(k)}$ is not n, then $\det(A) = 0$. Denote by $\mathcal{A}_n$ the set of all principal minors of order bn, such that every $A^{(k)}$ has order n. Then

$$c_{bn} = \sum_{A\in\mathcal{A}_n}\det(A) = (-1)^{n(b-1)}\sum_{A\in\mathcal{A}_n}\prod_{k=1}^{b}\det(A^{(k)}). \tag{2.8}$$

Theorem 2.4. *If $p > (d-e)(2d-1)$, then*

$$\operatorname{ord}_\pi(\det(A)) \ge ab(p-1)P_{u,e,d}(n+1)$$

for any $A \in \mathcal{A}_{a(n+1)}$.

Proof of Theorem 1.1. By Theorem 2.4 and (2.8), we have

$$\operatorname{ord}_\pi(c_{abn}) \ge ab(p-1)P_{u,e,d}(n).$$

Thus $\mathrm{NP}_{u,T}(f)$ lies above $P_{u,e,d}$ by Theorem 2.3. Note that $\mathrm{NP}_{u,m}(f) \ge \mathrm{NP}_{u,T}(f)$ by definition. Therefore, $\mathrm{NP}_{u,m}(f)$ also lies above $P_{u,e,d}$. □

2.3. Estimation on c_n.

Denote by

$$\phi(n) = \min\{x+y \mid dx+ey=n, x,y\in\mathbb{N}\} \in \mathbb{N}\cup\{+\infty\}.$$

Here the minimal element in $\emptyset$ is regarded as $+\infty$. For $i,j\in\mathbb{N}, k\in I_b$, define

$$\gamma_{(\frac{s_{k-1}}{q-1}+i,\frac{s_k}{q-1}+j)} = \pi^{\frac{s_k-s_{k-1}}{d(q-1)}+\frac{j-i}{d}}\gamma_{pi-j+u_{-k}}. \tag{2.9}$$

Then

$$\xi_w^{\sigma^{-1}}\gamma^{\sigma^{-1}}_{(\frac{s_{k-1}}{q-1}+i,\frac{s_k}{q-1}+j)} = \sum_{u\in I_a}\gamma_{(v,\frac{s_{k-1}}{q-1}+i),(w,\frac{s_k}{q-1}+j)}\xi_v$$

and

$$\begin{aligned} &\mathrm{ord}_\pi\left(\gamma_{(v,\frac{s_{k-1}}{q-1}+i),(w,\frac{s_k}{q-1}+j)}\right) \geq \mathrm{ord}_\pi\left(\gamma_{(\frac{s_{k-1}}{q-1}+i,\frac{s_k}{q-1}+j)}\right) \\ =&\frac{s_k-s_{k-1}}{d(q-1)}+\frac{j-i}{d}+\phi(pi-j+u_{-k}). \end{aligned} \tag{2.10}$$

Lemma 2.5. *For any $\tau\in S_n^*$ and integer t,*

$$\sum_{i=0}^{n}\phi(pi-\tau(i)+t)\geq d^{-1}\left(\frac{(p-1)n(n+1)}{2}+(n+1)t+(d-e)C_{t,n}\right).$$

Proof. We may assume that $pi-\tau(i)+t\in d\mathbb{N}+e\mathbb{N}$ for each i. One can easily show that

$$\phi(k)=d^{-1}\left(k+(d-e)\overline{e^{-1}k}\right)$$

and the minimum arrives at

$$(x,y)=\left(d^{-1}(k-e\overline{e^{-1}k}),\overline{e^{-1}k}\right).$$

Thus

$$\phi(pi-j+t)=d^{-1}\left(pi-j+t+(d-e)\overline{e^{-1}(pi-j+t)}\right). \tag{2.11}$$

The result then follows easily. ◻

Lemma 2.6. *Assume $a_i=a_{i+m}$ and $b_i=b_{i+m}$ for any $i\in I_{md}$. Then*

$$\max_{\tau\in S_{md}}\#\left\{i\in I_{md}\mid a_i\geq b_{\tau(i)}\right\}=d\max_{\sigma\in S_m}\#\left\{i\in I_m\mid a_i\geq b_{\sigma(i)}\right\}.$$

Proof. We may assume that there exists some k such that: $a_k\geq b_k$ and for any $i\neq k$, $b_i>a_k$ or $b_i\leq b_k$. Otherwise both sides should be zero. We may assume that $k=m$ for simplicity. Apply Lemma 2.2 d times, where $a_0=a_{mi},b_0=b_{mi}$ in i-th time, we get

$$\max_{\tau'\in S_{md}}\#\left\{i\in I_{md}\mid a_i\geq b_{\tau'(i)}\right\}=d+\max_{\tau}\#\left\{i\in I_{md}-m\mathbb{Z}\mid a_i\geq b_{\tau(i)}\right\},$$

where τ runs through permutations on $I_{md}-m\mathbb{Z}$. Since

$$\max_{\sigma'\in S_m}\#\left\{i\in I_m\mid a_i\geq b_{\sigma'(i)}\right\}=1+\max_{\sigma}\#\left\{i\in I_m-\{m\}\mid a_i\geq b_{\sigma(i)}\right\}$$

by Lemma 2.2, where σ runs through permutations on $I_m-\{m\}$, the result is reduced to

$$\max_{\tau}\#\left\{i\in I_{md}-m\mathbb{Z}\mid a_i\geq b_{\tau(i)}\right\}=d\max_{\sigma}\#\left\{i\in I_m-\{m\}\mid a_i\geq b_{\sigma(i)}\right\}.$$

Denote by $A_{(m-1)i+j}=a_{mi+j}$ and $B_{(m-1)i+j}=b_{mi+j},1\leq j\leq m-1$. Then $A_i=A_{i+m-1},B_i=B_{i+m-1}$ and the equation above becomes

$$\max_{\tau\in S_{(m-1)d}}\#\left\{i\in I_{(m-1)d}\mid A_i\geq B_{\tau(i)}\right\}=d\max_{\sigma\in S_{m-1}}\#\left\{i\in I_{m-1}\mid A_i\geq B_{\sigma(i)}\right\}.$$

The result then follows by induction on m. ◻

Lemma 2.7. *For any $i\in\mathbb{N}\times I_a$, we write $i=(i',i'')$. Then for any permutation τ on $I_n^*\times I_a$,*

$$\sum_{i\in I_n^*\times I_a}\phi(pi'-\tau(i)'+t)\geq\frac{a}{d}\left(\frac{(p-1)n(n+1)}{2}+(n+1)t+(d-e)C_{t,n}\right).$$

Proof. By Eq. (2.11), we only need to show that

$$\min_{\tau} \sum_{i\in I_n^*\times I_a} \overline{e^{-1}(pi-\tau(i)+t)} = aC_{t,n}.$$

By Proposition 2.1, it can be reduced to

$$\max_{\tau} \#\left\{i\in I_n^*\times I_a \mid R_{i',\alpha}+r_{\tau(i)',\alpha}\geq d\right\} = a\mathbf{C}_{t,n,\alpha}.$$

This follows from Lemma 2.6. □

Proof of Theorem 2.4. This proof is similar to [ZN21, Theorem 3.2]. Denote by $\mathcal{R}$ the set of indices of A and

$$\mathcal{R}^{(k)}\times\{k\} = \mathcal{R}\cap(\mathbb{N}\times I_a\times\{k\}),\quad \mathcal{R}^{(0)}=\mathcal{R}^{(b)}.$$

Then $\#\mathcal{R}^{(k)}=a(n+1)$,

$$A^{(k)} = \left(\gamma_{(v,\frac{s_{k-1}}{q-1}+i),(w,\frac{s_k}{q-1}+j)}\right)_{(i,v)\in\mathcal{R}^{(k-1)},(j,w)\in\mathcal{R}^{(k)}}$$

and

$$\det(A)=\prod_{k=1}^{b}\det(A^{(k)})=\sum_{\tau}\operatorname{sgn}(\tau)\prod_{i\in\mathcal{R}}\gamma_{i,\tau(i)},$$

where τ runs through permutations of $\mathcal{R}$ such that $\tau(\mathcal{R}^{(k-1)})=\mathcal{R}^{(k)}$. Here,

$$\operatorname{ord}_{\pi}\left(\prod_{i\in\mathcal{R}}\gamma_{i,\tau(i)}\right)\geq S^{\tau}_{\mathcal{R}}$$

by (2.10), where

$$\begin{aligned}S^{\tau}_{\mathcal{R}} &= \sum_{k=1}^{b}\sum_{i\in\mathcal{R}^{(k-1)}}\left(\frac{\tau(i)'-i'}{d}+\phi(pi'-\tau(i)'+u_{-k})\right)\\ &\geq d^{-1}\sum_{k=1}^{b}\sum_{i\in\mathcal{R}^{(k-1)}}\left((p-1)i'+(d-e)\overline{e^{-1}(pi'-\tau(i)'+u_{-k})}\right)\end{aligned}$$

by Eq. (2.11). By Lemma 2.7,

$$S^{\sigma}_{\mathcal{N}}\geq ab(p-1)P_{u,e,d}(n+1),$$

where $\mathcal{N}=I_n^*\times I_a\times I_b$. By (2.8), we only need to show that for any permutation τ of $\mathcal{R}\neq\mathcal{N}$ such that $\tau(\mathcal{R}^{(k-1)})=\mathcal{R}^{(k)}$, there is a permutation σ of $\mathcal{N}$ such that $\sigma(\mathcal{N}^{(k-1)})=\mathcal{N}^{(k)}$ and $S^{\tau}_{\mathcal{R}}\geq S^{\sigma}_{\mathcal{N}}$.

Assume $\#(\mathcal{R}\backslash\mathcal{N})=m$. Write $T=(\mathcal{N}\backslash\mathcal{R})\cup\tau^{-1}(\mathcal{R}\backslash\mathcal{N})$, then $\#T=2m$ and $\mathcal{N}\backslash T=\mathcal{N}\cap\tau^{-1}(\mathcal{N}\cap\mathcal{R})$. Thus $\tau(\mathcal{N}\backslash T)\subset\mathcal{N}$. Note that for $i\in\mathcal{R}\backslash\mathcal{N}, j\in\mathcal{N}\backslash\mathcal{R}$, $i'\geq n+1\geq j'+1$. We can choose a permutation σ of $\mathcal{N}$ such that $\sigma(\mathcal{N}^{(k-1)})=\mathcal{N}^{(k)}$ and $\sigma=\tau$ on $\mathcal{N}\backslash T$. Then

$$\begin{aligned}&d(S^{\tau}_{\mathcal{R}}-S^{\sigma}_{\mathcal{N}})\\ \geq&\left(\sum_{i\in\mathcal{R}\backslash\mathcal{N}}-\sum_{i\in\mathcal{N}\backslash\mathcal{R}}\right)(p-1)i'-\sum_{k=1}^{b}\sum_{i\in T\cap\mathcal{N}^{(k)}}(d-e)\overline{e^{-1}(pi'-\tau(i)'+u_{-k})}\\ \geq& m(p-1)-2m(d-e)(d-1)>0.\end{aligned}$$

The result then follows. □

3. The Newton polygons

Lemma 3.1. *The Newton polygon* $\mathrm{NP}_m(f)$ *lies over* $\mathrm{NP}_T(f)$. *Moreover, if the equality holds for one* m*, then it holds for all* m*.*

Proof. See [LW09, Theorem 2.3] and [LN11, Theorem 5.5]. □

Proof of Theorem 1.2. (1) Since $w(d+i)=1+w(i)$, both of $\mathrm{NP}_{u,m}(f)$ and $P_{u,e,d}$ across points $\big(di, H^{\infty}_{[0,d],u}(di)\big)$, we only need to show that $\mathrm{NP}_{u,m}(f)=P_{u,e,d}$ on $[1,d-1]$. By Lemma 3.1, we may assume that $m=1$.

Assume $0\le n\le d-2$. Recall that $S^{\circ}_{t,n}$ is the set of $\tau\in S^*_n$ such that

$$\#\left\{i\in I^*_n \mid R_{i,\alpha}+r_{\tau(i),\alpha}\ge d\right\}=\mathbf{C}_{t,n,\alpha}$$

and every $pi-\tau(i)+t\in d\mathbb{N}+e\mathbb{N}$. It's equivalently to say, the equality in Lemma 2.5 holds. Recall that

$$y^{\tau}_{t,i}=\overline{e^{-1}(pi-\tau(i)+t)},\quad x^{\tau}_{t,i}=\phi(pi-\tau(i)+t)-y^{\tau}_{t,i}.$$

Denote by m the right hand side in Lemma 2.5. Then we have

$$\det(\gamma_{pi-j+t})_{i,j\in I^*_n}\equiv\pi^m\sum_{\tau\in S^{\circ}_{t,n}}\mathrm{sgn}(\tau)\prod_{i=0}^{n}\lambda_{x^{\tau}_{t,i}}\lambda_{y^{\tau}_{t,i}}\hat{\lambda}^{y^{\tau}_{t,i}}$$

$$\equiv\pi^m\hat{\lambda}^{v_{t,n}}\sum_{\tau\in S^{\circ}_{t,n}}\mathrm{sgn}(\tau)\prod_{i=0}^{n}\frac{1}{x^{\tau}_{t,i}!y^{\tau}_{t,i}!}\mod \pi^{m+1},$$

where

$$v_{t,n}:=\sum_{i=0}^{n}y^{\tau}_{t,i}=\sum_{i=1}^{n}(R_{i,\alpha}+r_{i,\alpha})-d\mathbf{C}_{t,n,\alpha}$$

is independent on $\tau\in S^{\circ}_n$.

Recall that $S^{\tau}_{\mathcal{R}}>S^{\sigma}_{\mathcal{N}}$ in the proof of Theorem 2.4. Then modulo $\pi^{ab(p-1)P_{u,e,d}(n+1)+1}$, we have

$$\begin{aligned}c_{ab(n+1)}&=\sum_{A\in\mathcal{A}_{a(n+1)}}\det(A)\equiv\det\big((\gamma_{i,j})_{i,j\in\mathcal{N}}\big)\\&=\pm\mathrm{Nm}\left(\prod_{k=1}^{b}\det\left(\gamma_{(\frac{s_{k-1}}{q-1}+i,\frac{s_k}{q-1}+j)}\right)_{i,j\in I^*_n}\right)\\&=\pm\mathrm{Nm}\left(\prod_{k=1}^{b}\det(\gamma_{pi-j+u_k})_{i,j\in I^*_n}\right)\\&\equiv\pm\pi^{ab(p-1)P_{u,e,d}(n+1)}\mathrm{Nm}\left(\prod_{k=1}^{b}\hat{\lambda}^{v_{u_k,n}}h_{n,k}\right)\end{aligned}$$

by (2.8), (2.9), [LLN09, Lemma 4.4] and [LN11, Lemma 3.5]. Hence we get the first assertion by replacing π by π_1.

(2) Denote by t_k the minimal non-negative residue of $p^{-k}\mu$ modulo c. Then $u_k=\frac{t_{k+1}p-t_k}{c}$. Write $\mathbf{p}$ the minimal positive residue of p modulo cd and $p=cd\ell+\mathbf{p}$. Denote by

$$\mathbf{u}_k=\frac{t_{k+1}\mathbf{p}-t_k}{c},\ \mathbf{y}^{\tau}_{\mathbf{u}_k,i}=\overline{-e^{-1}(\mathbf{p}i-\tau(i)+\mathbf{u}_k)},\ \mathbf{x}^{\tau}_{\mathbf{u}_k,i}=\frac{\mathbf{p}i-\tau(i)+\mathbf{u}_k-e\mathbf{y}^{\tau}_{\mathbf{u}_k,i}}{d}.$$

Then

$$u_k = t_{k+1}d\ell + \mathbf{u}_k,\ y^\tau_{u_k,i} = \mathbf{y}^\tau_{\mathbf{u}_k,i},\ x^\tau_{u_k,i} = (ci + t_{k+1})\ell + \mathbf{x}^\tau_{\mathbf{u}_k,i}.$$

It's easy to see that $\mathbf{x}^\tau_{\mathbf{u}_k,i} < \mathbf{p}$ and $x^\tau_{u_k,i} < p$. Since

$$\mathbf{x}^\tau_{\mathbf{u}_k,i} \geq \frac{-n - e(d-1)}{d} > -e - 1,$$

we have $\mathbf{x}^\tau_{\mathbf{u}_k,i} \geq -e$. Note that $y^\tau_{t,i}$ does not depend on ℓ. Denote by

$$\begin{aligned} H_{\mu,c,\mathbf{p},e,d} = &\prod_{k=1}^{b}\prod_{n=0}^{d-2}\sum_{\tau\in S_n^\circ}\operatorname{sgn}(\tau)\prod_{i=1}^{n}(d-1)_{\left[d-1-\mathbf{y}^\tau_{\mathbf{u}_k,i}\right]} \times (cd)^{\mathbf{p}-1-\mathbf{x}^\tau_{\mathbf{u}_k,i}} \\ &\times \left(-\frac{\mathbf{p}(ci+t_{k+1})}{cd} + \mathbf{p} - 1\right)_{\left[\mathbf{p}-1-\mathbf{x}^\tau_{\mathbf{u}_k,i}\right]} \in \mathbb{Z}. \end{aligned} \tag{3.1}$$

Then

$$\begin{aligned} &H_{\mu,c,\mathbf{p},e,d} \\ \equiv &\prod_{k=1}^{b}\prod_{n=0}^{d-2}\sum_{\tau\in S_n^\circ}\operatorname{sgn}(\tau)\prod_{i=1}^{n}(d-1)_{\left[d-1-\mathbf{y}^\tau_{\mathbf{u}_k,i}\right]} \times (cd)^{\mathbf{p}-1-\mathbf{x}^\tau_{\mathbf{u}_k,i}} \\ &\times ((ci+t_{k+1})\ell + \mathbf{p} - 1)_{\left[\mathbf{p}-1-\mathbf{x}^\tau_{\mathbf{u}_k,i}\right]} \\ = &h_{u,e,d}\prod_{k=1}^{b}\prod_{n=0}^{d-2}\prod_{i=1}^{n}(d-1)!(cd)^{\mathbf{p}-1-\mathbf{x}^\tau_{\mathbf{u}_k,i}}\big((ci+t_{k+1})\ell + \mathbf{p} - 1\big)! \mod p \end{aligned}$$

Note that $d-1, (ci+t_{k+1})\ell + \mathbf{p} - 1 < p$. Thus

$$\mathrm{NP}_{u,m}(f) = \mathrm{NP}_{u,T}(f) = P_{u,e,d} \iff p \nmid H_{\mu,c,\mathbf{p},e,d}$$

for $p > (d-e)(2d-1)$. □

Proof of Corollary 1.3. Since $p \nmid H_{\mu,c,\mathbf{p},e,d}$, we have $H_{\mu,c,\mathbf{p},e,d} \neq 0$. Hence $p' \nmid H_{\mu,c,\mathbf{p},e,d}$ for any $p' > H_{\mu,c,\mathbf{p},e,d}$. Note that

$$\sum_{k=1}^{b} u_k = \frac{p-1}{c}\sum_{k=1}^{b} t_k,$$

thus $H^\infty_{[0,d],u}$ only depends on $\mu, c, \mathbf{p}, d$. Since

$$P_{u,e,d}(n) - H^\infty_{[0,d],u}(n) = \frac{d-e}{bd(p-1)}\sum_{k=1}^{b} C_{u_k,n-1} \leq \frac{(d-e)\overline{n}(d-1)}{d(p-1)}$$

tends to zero as p tends to infinity, the result then follows. □

Example 3.2. Assume that $p \equiv 1 \bmod d$ and $d \mid u_k$ for all k. Write $p = dk+1$ and $t = u_k$. Then

$$R_i := R_{i,0} = \overline{e^{-1}i}, \quad R_i := r_{i,0} = \overline{-e^{-1}i}, \quad \mathbf{C}_{t,n} = n, \quad S_n^\circ = \{1\}$$

and $x^1_{t,i} = \frac{(p-1)i+t}{d}, y^1_{t,i} = 0$. Since

$$h_{n,k} = \left(\prod_{i=0}^{n}\left(\frac{(p-1)i + u_k}{d}\right)!\right)^{-1} \in \mathbb{Z}_p^\times,$$

we obtain that the Newton polygons coincide $H^\infty_{[0,d],u}$.

4. The case $e = d-1$

If $pi-\tau(i)+t \notin d\mathbb{N}+e\mathbb{N}$ for some i, then $x^{\tau}_{t,i}<0$. Set $1/k! = 0$ for negative integer k. Then

$$h_{n,k} = \sum_{\tau\in S^{\bullet}_{u_k,n}} \mathrm{sgn}(\tau)\prod_{i=1}^{n}\frac{1}{x^{\tau}_{u_k,i}!y^{\tau}_{u_k,i}!},$$

where $S^{\bullet}_{t,n}$ the set of $\tau\in S^{*}_{n}$ such that the size of $\left\{i\in I^{*}_{n}\mid R_{i,\alpha}+r_{\tau(i),\alpha}\geq d\right\}$ is $C_{t,n,\alpha}$.

Lemma 4.1. *Denote by* $c(j)=(-\alpha j+\beta)_{[j]}$.

(1) If $u_i=\alpha v_i+\beta$ *for any* i, *then the matrix*

$$\left((u_i)_{[j]}\cdot(v_i+n)_{[n-j]}\right)_{0\leq j\leq n}\implies\left(c(j)v_i^{n-j}\right)_{0\leq j\leq n} \tag{4.1}$$

by third elementary column transformations.

(2) If $u_i\equiv\alpha v_i+\beta \bmod p$ *for any* i, *then* (4.1) *holds by third elementary column transformations, modulo* p.

Proof. (1) Write

$$(\alpha x+\beta)_{[j]}=\sum_{t=0}^{j}c_t(j)\cdot(x+j)_{[t]}\,,$$

then $c_0(j)=c(j)$ and

$$\begin{aligned}&(u_i)_{[j]}\cdot(v_i+n)_{[n-j]}\\ =&\sum_{t=0}^{j}c_t(j)\cdot(v_i+j)_{[t]}\cdot(v_i+n)_{[n-j]}\\ =&\sum_{t=0}^{j}c_t(j)\cdot(v_i+n)_{[n-j+t]}\,.\end{aligned}\tag{4.2}$$

Hence by third elementary column transformations,

$$\left((u_i)_{[j]}\cdot(v_i+n)_{[n-j]}\right)\implies\left(c(j)\cdot(v_i+n)_{[n-j]}\right)\implies\left(c(j)v_i^{n-j}\right).$$

(2) In this case, (4.2) holds modulo p. The result then follows easily. ☐

Proof of Theorem 1.4. Since $p>c(d^2-d+1)$, we have $p>(d-e)(2d-1)$. Denote by $t=u_k$ and t_k the minimal non-negative residue of $p^{-k}\mu$ modulo c. Then $t=\frac{t_{k+1}p-t_k}{c}$. If $c>1$, then $t\geq\frac{p-(c-1)}{c}\geq d(d-1)$ and $t<\frac{(c-1)p}{c}\leq p-d(d-1)$. If $c=1$, then $t=0$.

Assume that $0\leq n\leq d-2$. Denote by

$$R_i=R_{i,t}=\overline{e^{-1}(pi+t)}=\overline{-pi-t}=-pi-t+\ell_i d$$

and

$$r_i=r_{i,t}=\overline{-e^{-1}i}=\bar{i}.$$

Then

$$\{d-r_i\mid i\in I^{*}_{n}\}=\{d,d-1,\dots,d-n\}\,.$$

We have

$$\mathbf{C}_{t,n}=\#\left\{i\in I^{*}_{n}\mid R_i\geq d-n\right\}$$

and
$$S_n^{\bullet} = \{\tau \in S_n^* \mid R_i + \tau(i) \geq d \text{ for } R_i \geq d-n\}.$$
For $R_i < d-n$, we have $R_i + \tau(i) < d$ and
$$x_{t,i}^{\tau} = pi + t - \ell_i e - \tau(i), \quad y_{t,i}^{\tau} = -pi - t + \ell_i d + \tau(i);$$
for $R_i \geq d-n$, we have $R_i + \tau(i) \geq d$ and
$$x_{t,i}^{\tau} = pi + t - \ell_i e + e - \tau(i), \quad y_{t,i}^{\tau} = -pi - t + \ell_i d - d + \tau(i).$$
If $\tau \notin S_n^{\bullet}$, there is i such that $y_{t,i}^{\tau} < 0$ or $x_{t,i}^{\tau} < 0$. Denote by
$$(u_i, v_i) = \begin{cases} (pi + t - \ell_i e, -pi - t + \ell_i d), & \text{if } R_i < d-n; \\ (pi + t - \ell_i e + e, -pi - t + \ell_i d - d), & \text{if } R_i \geq d-n. \end{cases}$$
Then
$$h_{n,k} = \det\left(\frac{1}{(u_i - j)!(v_i + j)!}\right).$$
Apply Lemma 4.1(2) with $\alpha = -d^{-1}e, \beta = t(1 - d^{-1}e)$, we obtain that
$$\begin{aligned} & h_{n,k} \cdot \prod_{i=0}^{n} u_i! \cdot (v_i + n)! \\ \equiv & \prod_{j=0}^{n} \left(d^{-1}e(j-t) + t\right)_{[j]} \cdot \det\left(v_i^{n-j}\right) \\ \equiv & \prod_{j=0}^{n} \left(d^{-1}e(j-t) + t\right)_{[j]} \cdot \prod_{0 \leq i < j \leq n} (v_i - v_j) \mod p. \end{aligned}$$
If $R_i < d-n$, then $v_i = R_i \geq 0$; if $R_i \geq d-n$, then $v_i + n = R_i - d + n \geq 0$. Hence $0 \leq v_i + n \leq d-1$ are different and $(v_i + n)!, (v_i - v_j) \in \mathbb{Z}_p^{\times}$ if $i \neq j$. Note that $u_i = \ell_i - R_i$ or $\ell_i - R_i + e$. When $c = 1$, we have $t = R_0 = \ell_0$, $u_0 = 0$ or e, and for $i \geq 1$,
$$u_i \geq \ell_i - R_i \geq \frac{pi + t}{d} - d + 1 \geq \frac{p}{d} - d + 1 \geq 0.$$
When $c > 1$, we have
$$u_i \geq \ell_i - R_i \geq \frac{pi + t}{d} - d + 1 \geq \frac{t}{d} - d + 1 \geq 0.$$
Meanwhile,
$$u_i \leq \ell_i - R_i + e = \frac{pi + t - (d-1)R_i + de}{d} \leq \frac{p(d-2) + t + de}{d} < p,$$
hence $u_i! \in \mathbb{Z}_p^{\times}$.

For any $0 \leq k \leq j-1$, we have
$$0 < e(j-t) + d(t-k) = d(j-k) + t - j \leq (d-1)j + p - d(d-1) < p,$$
which means that $p \mid \left(d^{-1}e(j-t) + t\right)_{[j]}$. Hence $h_{n,k} \in \mathbb{Z}_p^{\times}$. □

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